Green's Function Technique for Solving Anisotropic Electrostatic Field Problems

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***Abstract*—The validity of the reciprocity relation satisfied by the Green's function is discussed for a multidielectric region with inhomogeneous anisotropic media. A theoretical proof on the use of the Green's function technique for solving boundary-value problems is discussed for general cases.**

I. Introduction

THE PARAMETERS of a microstrip line based on a TEM approximation can be calculated from the line capacitance between the conductors. The line capacitance can be obtained by using the Green's function satisfying the boundary conditions [1]–[4]. However, in most previous works, isotropic substrate materials are treated. Recently the effect of anisotropy of the sapphire on the quasi-static characteristics of microstrip lines have been investigated by the finite difference method [5].

The impedance and velocity matching, which are principal limiting factors of the electrooptic light modulator bandwidth, have been studied [6]–[8]. The modulator is made of an anisotropic crystal. The parameters can be derived from the line capacitance. The latter can be calculated by using the Green's function. The line capacitance of a thin-film electrooptic light modulator with parallel-strip electrodes was calculated by the variational technique using the Green's function in the Fourier-transformed domain [8]. In these problems involving anisotropic materials, various properties of field functions are needed.

This paper reports that the reciprocity relation satisfied by the Green's function is valid for a multidielectric region with inhomogeneous anisotropic media by an extension of the reciprocity relation for isotropic media [9] and shows the theoretical proof on what region the Green's function is determined and how to apply the Green's function in solving boundary-value problems. Such theoretical proof for a multidielectric region with anisotropic media has not appeared in literature. Numerical analyses of the microstrip and the modulator by using this Green's function technique are reported in another paper [10].

II. Reciprocity Relation for Green's Function

In the three-dimensional space, consider the region R' containing only the dielectric medium whose permittivity

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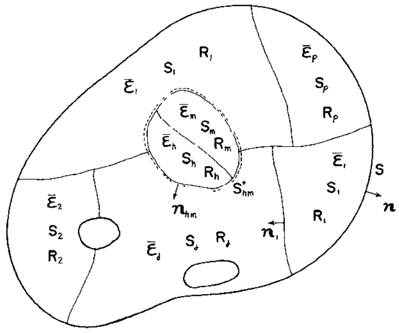

Fig. 1. A p-dielectric region R with inhomogeneous anisotropic media.

$\bar{\epsilon}(r)$ is a tensor of position r and continuous with continuous first-order partial derivatives. Let $u(r)$ and $w(r)$ be two scalar functions of position in the region R'. Applying the divergence theorem to the two vector functions, $\bar{\epsilon}(r)\cdot\{u(r)\nabla w(r)\}$ and $\bar{\epsilon}(r)\cdot\{w(r)\nabla u(r)\}$, and subtracting the latter result from the former, we obtain Green's theorem

$$\iiint_{R'} \left[u(r)\nabla\cdot\{\bar{\epsilon}(r)\cdot\nabla w(r)\} - w(r)\nabla\cdot\{\bar{\epsilon}(r)\cdot\nabla u(r)\}\right] dv + \iiint_{R'} \left[\{\bar{\epsilon}(r)-\bar{\epsilon}^T(r)\}\cdot\nabla w(r)\right]\cdot\nabla u(r)\, dv = \iint_{S'} \left[\bar{\epsilon}(r)\cdot\{u(r)\nabla w(r) - w(r)\nabla u(r)\}\right]\cdot n\, da \tag{1}$$

where n denotes the unit outward normal vector to the surface S' enclosing the region R' and T denotes the transpose.

Next, consider a p-dielectric region R as shown in Fig. 1 in the three-dimensional space: there is no need to consider S_{hm}^{+} and n_{hm}, shown in Fig. 1. They will be used in the next section. The region R enclosed by the surface S is composed of the regions R_i $(i=1,2,\cdots,p)$. Let the region R_i be filled with the inhomogeneous and anisotropic dielectric medium of permittivity $\bar{\epsilon}_i(r)$, which is a tensor with continuous first-order partial derivatives. Also assume that R_i is bounded by the surface S_i which is piecewise smooth. The source point (x_0,y_0,z_0) will be designated by r_0, the observation point (x,y,z) by r, and the three-dimensional delta function by $\delta(r-r_0)$.

The Green's function $G(r;r_0)$ is defined as a solution of the three-dimensional inhomogeneous partial differential equation

$$\nabla\cdot\{\bar{\epsilon}(r)\cdot(\nabla G)\} = -\delta(r-r_0) \tag{2}$$

subject to the homogeneous boundary condition

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$$\alpha(r)\{\bar{\epsilon}(r)\cdot(\nabla G)\}\cdot n+\beta(r)G=0, \quad \text{on } S \tag{3}$$

and the boundary conditions at the interfaces $S_i \cap S_j$ $(i,j=1,2,\cdots,p$, where $i\neq j)$

$$G|_{S_i\cap S_j, r\in S_i}=G|_{S_j\cap S_i, r\in S_j} \tag{4}$$

$$\{\bar{\epsilon}_i(r)\cdot(\nabla G)\}\cdot n_i|_{S_i\cap S_j, r\in S_i}=-\{\bar{\epsilon}_j(r)\cdot(\nabla G)\}\cdot n_j|_{S_j\cap S_i, r\in S_j} \tag{5}$$

where

$r_0 \quad = r_i$, $G\equiv G(r;r_i)$ when $r_0\in R_i$ $(i=1,2,\cdots,p)$
$\bar{\epsilon}(r) \quad =\bar{\epsilon}_i(r)$ when $r\in R_i$ $(i=1,2,\cdots,p)$,
n unit outward normal vector to S,
n_i unit outward normal vector to S_i $(i=1,2,\cdots,p)$.

The condition (3) can be classified into the following three cases: the Dirichlet problem

$$\alpha(r)=0, \quad \beta(r)\neq 0, \quad \text{on } S, \tag{6}$$

the Neumann-like problem

$$\alpha(r)\not\equiv 0, \quad \beta(r)=0, \quad \text{on } S, \tag{7}$$

and the mixed problem

$$\alpha(r)\not\equiv 0, \quad \beta(r)\neq 0, \quad \text{on } S. \tag{8}$$

Replacing $u(r)$ by $G(r;r_i)$, and $w(r)$ by $G(r;r_k)$ in Green's theorem (1) and using the method similar to that described in [9], we can derive the reciprocity relation

$$G(r_i;r_k)=G(r_k;r_i) \tag{9}$$

if

$$\bar{\epsilon}^T(r)=\bar{\epsilon}(r). \tag{10}$$

Therefore, we find that the reciprocity relation (9) satisfied by the Green's function for the inhomogeneous partial differential equation (2) is valid for the boundary-value problems (6)–(8).

III. Green's Function Technique for Boundary-Value Problems

Here we derive the theoretical proof on what region the Green's function is determined and how to apply the Green's function in solving boundary-value problems. Consider the region R_{BVP} given by removing the subregion $R_{hm}(=R_h\cup R_m)$ with the surface $S_{hm}^+(S_{hm}=S_h\cup S_m-S_h\cap S_m-S_m\cap S_h)$ from the region R of Fig. 1 and bounded by the surface $S_{BVP}(=S\cup S_{hm}^+)$. In the region R_{BVP}, we consider the following boundary-value problem with the homogeneous boundary condition on S and with the nonhomogeneous boundary condition on S_{hm}^+ as a general example:

$$\nabla\cdot\{\bar{\epsilon}(r)\cdot(\nabla\phi)\}=0, \quad \text{in } R_{BVP} \tag{11}$$

subject to the homogeneous boundary condition

$$\alpha(r)\{\bar{\epsilon}(r)\cdot(\nabla\phi)\}\cdot n+\beta(r)\phi=0, \quad \text{on } S \tag{12}$$

and the boundary conditions at the interfaces $S_i\cap S_j$ $(i,j=1,2,\cdots,p$; where $i\neq j$ and $i,j\neq h,m)$

$$\phi|_{S_i\cap S_j, r\in S_i}=\phi|_{S_j\cap S_i, r\in S_j} \tag{13}$$

$$\{\bar{\epsilon}_i(r)\cdot(\nabla\phi)\}\cdot n_i|_{S_i\cap S_j, r\in S_i}=-\{\bar{\epsilon}_j(r)\cdot(\nabla\phi)\}\cdot n_j|_{S_j\cap S_i, r\in S_j} \tag{14}$$

and the nonhomogeneous boundary condition on the surface S_{hm}^+

$$\xi(r)\{\bar{\epsilon}(r)\cdot(\nabla\phi)\}\cdot n_{hm}+\eta(r)\phi=\zeta(r), \quad \text{on } S_{hm}^+. \tag{15}$$

Now we consider the media which satisfy (10) in R_{BVP} and R. We choose the solution of (2) subject to the boundary conditions (3)–(5) in the region R of Fig. 1 as the Green's function for the above boundary-value problem. We derive the potential $\phi(r_j)$ at an arbitrary point r_j in an arbitrary subregion R_j in R_{BVP} by using the Green's function $G_j\equiv G(r;r_j)$. Replacing $u(r)$ by $\phi(r)$, and $w(r)$ by G_j in Green's theorem (1) for the media satisfying (10), we obtain the following equation for the region R_k $(k=1,2,\cdots,p;$ where $k\neq h,m)$:

$$\iiint_{R_k}[\phi\nabla\cdot\{\bar{\epsilon}_k(r)\cdot(\nabla G_j)\}-G_j\nabla\cdot\{\bar{\epsilon}_k(r)\cdot(\nabla\phi)\}]\,dv$$
$$=\iint_{S_k}[\bar{\epsilon}_k(r)\cdot(\phi\nabla G_j)-\bar{\epsilon}_k(r)\cdot(G_j\nabla\phi)]\cdot n_k\,da. \tag{16}$$

Summing (16) over all k, substituting (2) and (11) into those volume integrals, and rearranging surface integrals by using (3)–(5) and (12)–(14), we obtain

$$\phi(r_j)=-\sum_{k=1,k\neq h,m}^{p}\iint_{S_k\cap(S_h\cup S_m)}[\bar{\epsilon}_k(r)\cdot(\phi\nabla G_j)$$
$$-\bar{\epsilon}_k\cdot(G_j\nabla\phi)]\cdot n_k\,da. \tag{17}$$

The right-hand side of (17) can be expressed in terms of a surface integral on the surface S_{hm}^+. Replacing r_j with r, and r with r_0, and using the reciprocity relation (9), we can express the potential at an arbitrary point r in R_{BVP} by the surface integral on S_{hm}^+ as follows:

$$\phi(r)=\iint_{S_{hm}^+}[\bar{\epsilon}(r_0)\cdot\{\phi(r_0)\nabla_0 G(r;r_0)\}$$
$$-\bar{\epsilon}(r_0)\cdot\{G(r;r_0)\nabla_0\phi(r_0)\}]\cdot n_{hm}\,da_0 \tag{18}$$

where $\bar{\epsilon}(r_0)=\bar{\epsilon}_k(r_0)$ when $r_0\in R_k$ $(k=1,2,\cdots,p;$ where $k\neq h,m)$, and ∇_0 and da_0 designate differentiation and integration with respect to x_0, y_0, z_0, respectively. It can be shown by using (2)–(5) that this potential $\phi(r)$ satisfies (11)–(14).

Therefore, the boundary-value problem with the nonhomogeneous boundary condition (15) on the surface S_{hm}^+ can be solved as the surface integral (18) on S_{hm}^+ by the use of the Green's function for the region given by adding the subregion which is bounded by S_{hm} and which can be divided into any subregions with any media satisfying (10).

Now, if the nonhomogeneous boundary condition on S_{hm}^+ is

$$\phi=V_0, \quad \text{on } S_{hm}^+ \tag{19}$$

(18) becomes the integral of (20). This is because the first term in integrand vanishes after integration on account of (1), (2), (4), and (5):

$$\phi(r)=\iint_{S_{hm}^{+}}\sigma(r_0)G(r;r_0)\,da_0 \tag{20}$$

where

$$\sigma(r_0)=\left[\bar{\epsilon}(r_0)\cdot\{-\nabla\phi(r_0)\}\right]\cdot n_{hm} \tag{21}$$

and the unknown charge distribution $\sigma(r_0)$ is determined by (19) and (20).

IV. Conclusion

The validity of the reciprocity relation satisfied by the Green's function has been discussed. The theoretical proof on the use of the Green's function technique has been discussed for general cases. The reciprocity relation and the Green's function technique will be useful for problems involving the Green's function of the form discussed here.

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